Table 1 Pretriangularized equations with forces as unknowns

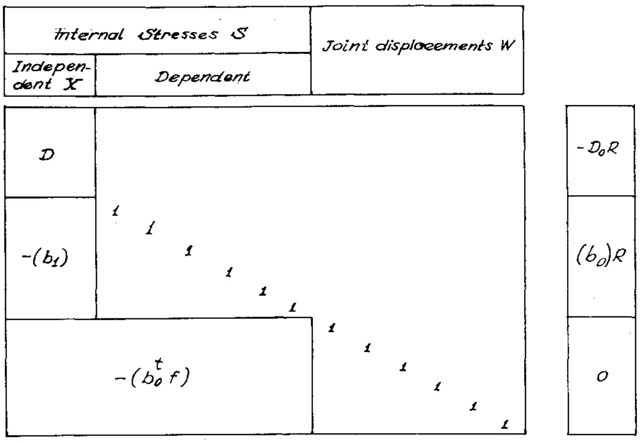

Table 2 Pretriangularized equations with displacements as unknowns

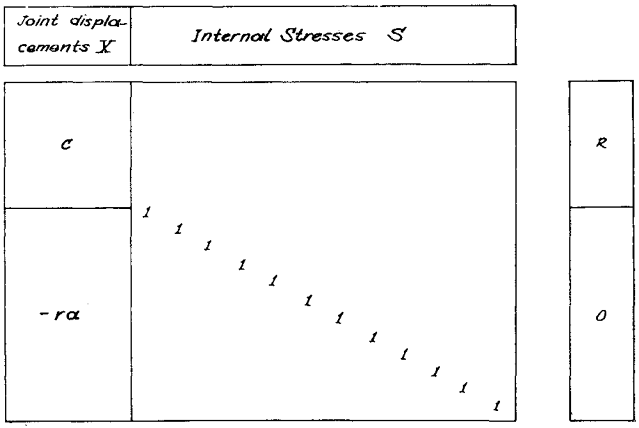

Table 3 Pretriangularized equations of the example

P_3	P_4	P_1	P_2	P_5	P_6	q_1a	q_2a	u_1	u_2	u_3	u_4	v_1	v_2	v_3	v_4	
42	-15.33															4
-15.33	16															2
1		1														1
	1		1													1
1				1												
-2	1				1											
1						1										
-1	1						1									
		-2	-1					1								1
		-1	-1						1							
-2	-1									1						
-1	-1										1					
				2								1				
					2								1			
				1										1		
					1										1	

If joint displacements are considered as unknowns, the response of the structure to external loads applied at the joints is given in Ref. 5 as

$$CX = R \tag{4}$$

and

$$S = raX \tag{5}$$

Equations (4) and (5) can be written jointly in Table 2.

Argyris' equations, put in the form of Tables 1 and 2, are exactly the equations of Klein when the ideal pretriangularization is attained, a situation in which there exists always a group of equations that are not pretriangularized. This group of equations is constituted in Table 1 by Eq. (3) and in Table 2 by Eq. (4).

Example

Consider the first example from Ref. 2. Taking as redundancies the internal stresses P_3 and P_4, the pretriangularized equations as in Table 1 are, for this case, given in Table 3. Solving the equations from Table 3, the solution given in Ref. 2 is obtained. Incidentally, there is a minor difference: in Ref. 2, with three decimals, $P_3 = 0.226$ and $P_4 = 0.344$. From Table 3, $P_3 = 0.216$ and $P_4 = 0.334$. This difference obviously is due to the fact that in Ref. 2 a matrix of order 16 was inverted and here a matrix of order 2.

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Equivalence between Chemical-Reaction and Volume-Viscosity Effects in Linearized Nonequilibrium Flows

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In the theory of sound propagation in a relaxing medium, it is a well-established fact that, for wave frequencies much smaller than the relaxation frequency, the relaxation process itself can be considered as having the same effect as a volume viscosity.[1] It is interesting to determine under what conditions the same statement would hold for steady nonequilibrium flows.

It is the purpose of this note to show that this happens for "linearized flows" when the ratio between a macroscopic characteristic time t_M and a suitably defined chemical characteristic time is much greater than one, i.e., near-equilibrium conditions. In these conditions, the basic equations for the linearized motion of a reacting medium are shown to reduce to those pertinent to an equivalent motion, at a Reynolds number defined in terms of appropriate thermodynamic derivatives, of an inert but viscous medium.

Assume the chemical affinity A, the specific volume v, and the entropy per unit mass of the mixture, s, as the basic set of independent thermodynamic variables. The appropriate thermodynamic potential ψ is the first-order Lagrange transform[2, 3] of the specific energy e with respect to the progress variable of the reaction ξ:

$$\psi = e - \xi A = \psi(s,v,A) \tag{1}$$

and the Gibbs relation

$$d\psi = Tds - pdv - \xi dA \tag{2}$$

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defines the conjugate intensive parameters T (temperature), p (pressure), and ξ as the derivatives of $\psi(s,v,A)$ with respect to (s,v,A), respectively.

If a superscript 0 indicates uniform equilibrium values and a superscript 1 indicates the small perturbations caused by initial and/or boundary conditions, the pertinent linearized equations read

$$\begin{aligned} D\xi'/Dt &= -L^0A' \qquad & Ds'/Dt &= 0 \\ Dv'/Dt &= v^0\nabla\cdot\mathbf{V}' \qquad & (D\mathbf{V}'/Dt) + v^0\nabla p' &= 0 \end{aligned} \tag{3}$$

where $D/Dt = (\partial/\partial t) + \mathbf{V}^0\cdot\nabla$, and $\mathbf{V}$ is the velocity vector.

The phenomenological coefficient L^0 is positive, due to the positive character of the entropy production. System (3) is to be implemented by the relations giving the dependent unknowns p' and ξ' as linear combinations of the fundamental set (s', v', A'). These relations are to be obtained from the two state equations $p = p(s,v,A)$ and $\xi = \xi(s,v,A)$.

Accounting for Eq. (3) and for the fact that $s' = 0$, one gets

$$-p' = \psi_{vA}{}^0A' + \psi_{vv}{}^0v' \tag{4a}$$

$$A' = (1/L^0)\psi_{AA}{}^0(DA'/Dt) + (1/L^0)\psi_{Av}{}^0(Dv'/Dt) \tag{4b}$$

where the subscripts indicate the partial derivatives of ψ computed at the equilibrium conditions of the basic flow $(A^0 = 0)$.

By definition [see also Eqs. (1) and (2)],

$$\begin{aligned} \psi_{vv}{}^0 &= -[(\partial p/\partial v)]_{A=0} = (1/v^{02})a_e{}^{02} > 0 \\ -\psi_{AA}{}^0 &= [(\partial\xi/\partial A)]_{A=0} = 1/e_{\xi\xi}{}^0 > 0 \end{aligned} \tag{5}$$

where $a_e{}^0$ is the equilibrium speed of sound pertinent to the basic flow, $e_{\xi\xi}{}^0$ is the second derivative of the specific energy $e(\xi,s,v)$ computed at equilibrium, and the inequalities follow from the thermodynamic stability conditions. Equation (5) permits definition of a "chemical-relaxation time" $(1/\tau) = (1/L^0e_{\xi\xi}{}^0)$ as an essentially positive quantity. Equation (4b) thus can be written as

$$A' = -(1/\tau)(DA'/Dt) + (1/\tau)(\psi_{Av}{}^0e_{\xi\xi}{}^0)(Dv'/Dt) \tag{6}$$

When the first term on the right-hand side of Eq. (6) can be neglected, one obtains

$$A' \cong (\psi_{Av}{}^0/L^0)(Dv'/Dt) \tag{7}$$

and subsequent substitution into Eq. (4a) yields

$$p' = -[(a_e{}^0)^2/v^{02}]v' - [v^0(\psi_{Av}{}^0)^2/L^0]\nabla\cdot\mathbf{V}' \tag{8}$$

thus showing that the effects of chemical reaction can be assimilated to those due to an effective coefficient of volume viscosity defined by

$$\eta_v{}^0 = [v^0(\psi_{Av}{}^0)^2/L^0] = (v^0/L^0)[(\partial\xi/\partial v)]_{A=0}{}^2 > 0 \tag{9}$$

For wave propagation in a medium at rest $(\mathbf{V}^0 = 0)$, the approximate relation (7) is valid when the ratio $\omega/\tau \ll 1$, where ω is the wave frequency. This is seen immediately by taking the Fourier transform (subscript F) of Eq. (6)

$$A_F' = (i\omega/\tau)A_F' - (i\omega/\tau)(\psi_{Av}{}^0e_{\xi\xi}{}^0)v_F'$$

where i is the imaginary unit. For $(\omega/\tau) \ll 1$, this relation reduces to

$$A_F' \cong -(i\omega/\tau)(\psi_{Av}{}^0e_{\xi\xi}{}^0)v_F'$$

which is nothing but the Fourier transform of Eq. (7). This essentially proves that the rate of change of A' can be neglected with respect to A' itself in Eq. (6) when the ratio between the pertinent characteristic time associated with this rate of change and $1/\tau$ is sufficiently small. It follows, then, rather straightforwardly, that Eq. (7) also will be valid for steady flows provided $[\mathbf{V}_r/\tau l_r] \ll 1$ (where $\mathbf{V}_r$ and l_r are suitable reference velocity and length), that is, when the macroscopic characteristic time $l_r/\mathbf{V}_r$ is much larger than the chemical one.

In this case the wave equation, obtained by scalar multiplication of the last of Eqs. (3) by $\mathbf{V}^0$ and substitution of Dp'/Dt from Eq. (8), is, accounting for Eq. (9),

$$[D(\mathbf{V}^0\cdot\mathbf{V}')/Dt] - a_e{}^{02}\nabla\cdot\mathbf{V}' - \eta_v{}^0(D/Dt)(\nabla\cdot\mathbf{V}') = 0$$

or, in terms of nondimensional quantities,

$$(D\mathbf{V}^0\cdot\mathbf{V}'/Dt) - a_e{}^{02}\nabla\cdot\mathbf{V}' = (1/Rev)\mathbf{V}^0\cdot\nabla(\nabla\cdot\mathbf{V}')$$

where $Rev = (\mathbf{V}_rl_r/v^0\eta_v{}^0)$ is a Reynolds number referred to the equivalent volume-kinematic viscosity $(v^0\eta_v{}^0)$ [Eq. (9)]. This form of the wave equation justifies and, at the same time, defines the limits of the use of singular-perturbation and/or boundary-layer-type techniques in the solution of nonequilibrium flows.

The approach could be extended to flows in which more chemical and/or relaxing processes occur. Much as in the problem of acoustical-wave propagation,[1] one could consider all those processes with relaxation times $1/\tau_1$, much smaller than the relevant convective characteristic time t_M, as contributing to an effective volume viscosity and thus treat explicitly as such only the processes with relaxation times of the same order as t_M. Detailed derivation of the wave equation for these cases will be presented in a future note.

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Location of the Normal Shock Wave in the Exhaust Plume of a Jet

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A METHOD-OF-CHARACTERISTICS program for calculating the exhaust plume flow field of a single axisymmetric jet recently has been completed.[1] The program assumes inviscid flow, with no mixing along the jet boundary.

Figure 1 shows a plot obtained from this program of the boundary and intercepting shock wave shapes for a jet exhausting into still air. As can be seen, the shock wave suddenly becomes normal at a point downstream. This note presents a new method for determining the location of this normal shock.

An approximate method previously suggested by Adamson and Nicholls[2] stated that the axial location of the normal shock was that point at which the static pressure behind the shock was equal to the receiver pressure. However, in actuality the subsonic flow behind the shock may accelerate to supersonic velocities and then pass through a series of weaker shocks.[3, 4] If this occurs, the preceding method can hold only for the last shock in the series.

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